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**BOUNDARY DYNAMICS OF HIGHER  
DIMENSIONAL CHERN-SIMONS  
GRAVITY**

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We review the relevance to the black hole entropy problem of boundary dynamics in Chern-Simons gravity. We then describe a recent derivation of the action induced on the four dimensional boundary in a five dimensional Chern-Simons gravity theory with gauge invariant, anti-deSitter boundary conditions.

## 1 Introduction

AdS spacetimes are under a great deal of scrutiny. One reason is their role in recent attempts to understand the microscopic source of black hole entropy, as given by the Bekenstein-Hawking formula:

$$S_{bh} = \frac{kA}{4G\hbar} \quad (1)$$

A detailed understanding of the origins of this formula is still lacking, but it is clear that it provides a remarkable mixture of geometry (the horizon area  $A$ ), gravity (Newton's constant,  $G$ ), quantum mechanics (Planck's constant  $\hbar$ ) and thermodynamics (Boltzmann's constant  $k$ ). It is widely believed that the explanation for this formula need not necessarily be tied to any particular theory of gravity, or its microscopic origin (such as string theory). Instead a correct explanation might apply equally well to any gravity theory that admits black hole solutions.

One particularly elegant attempt at such a universal explanation was proposed in the mid-nineties ([1],[2]). The idea was that black hole boundary conditions caused gauge/diffeomorphism modes to become physical along the horizon, in analogy to the edge currents that appear in the Chern-Simons

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description of the Quantum Hall Effect. A concrete implementation of this proposal was given by Carlip [1], who considered Einstein gravity in 2+1 dimensions with negative cosmological constant. This theory has a gauge theory formulation [3] as a Chern-Simons theory with group  $SO(2, 2) \sim SL(2, R) \times SL(2, R)$ . The corresponding action is:

$$S_{CS}^{(3)} = \int_{M^3} Tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \quad (2)$$

The solutions to this theory describe spacetimes with constant negative curvature. One such solution is the BTZ black hole [4] which can be obtained from 2+1 dimensional AdS spacetime by making suitable identifications. Carlip's argument went as follows: if one treats the black hole event horizon as a boundary then it is necessary to add to the action a surface term at the horizon in order to make the variational principle well defined. For example, in the case of the Euclidean BTZ black hole, the existence of a horizon requires, in holomorphic coordinates  $(z, \bar{z})$ , that  $A_z$  be fixed on the boundary (i.e. the horizon). The appropriate boundary term at the horizon is:

$$S_{bound} = \int_{\partial M^3} Tr(A_z A_{\bar{z}}) \quad (3)$$

The resulting total action  $S_{total} = S_{CS}^{(3)} + S_{bound}$  is not gauge invariant. Under the transformation  $A = g^{-1}\tilde{A}g + g^{-1}dg$ , it changes by

$$\delta S_{total} = S_{WZW2}[g, \tilde{A}_z] \quad (4)$$

where

$$S_{WZW2}[g, \tilde{A}_z] = \frac{1}{4\pi} \int_{\partial M} Tr(g^{-1}\partial_z g g^{-1}\partial_{\bar{z}}g - 2g^{-1}\partial_{\bar{z}}g \tilde{A}_z) + \frac{1}{12\pi} \int_M Tr(g^{-1}dg)^3 \quad (5)$$

This is the action for the gauged, chiral Wess-Zumino-Witten action in two dimensions ( $WZW_2$ ) [5]. More correctly, an independent chiral  $WZW_2$  action emerges for each copy of  $SL(2, R)$ . Carlip interpreted this as indicating that black hole boundary conditions caused certain gauge modes (or diffeomorphism modes in the geometrical theory) on the boundary to become dynamical. By quantizing the  $WZW_2$  boundary theory and counting states using methods in conformal field theory Carlip [1] was able to derive the correct Bekenstein-Hawking entropy for the BTZ black hole.

An interesting, but to some extent puzzling, variation of this scenario was introduced by A. Strominger [6]. He started from an old result of Brown and

Henneaux [7] who showed that 2+1 AdS spacetime contained an asymptotic set of symmetries consisting of a pair of Virasora algebras. Note that the original Brown and Henneaux paper only derived the algebra. It was not until much later that Coussart, Henneaux and van Driel [8] derived the boundary action and corresponding dynamics that give rise to this algebra. The action was that of the  $WZW_2$  model, suitably restricted to give a Liouville theory on the boundary. Strominger applied the Cardy formula to the central charge derived by Brown and Henneaux and was able to count the asymptotic density of states for boundary conditions at infinity consistent with the presence of a black hole. Remarkably, Strominger's calculation yielded precisely the Bekenstein-Hawking entropy of the black hole. It therefore seems that one could count black hole states either at infinity or the horizon.

The question we would like to address is: what happens in higher dimensions? The present discussion is based to a large extent on work published in [9].

## 2 Chern-Simons Gravity in $2n+1$ Dimensions

The Chern-Simons action Eq.(2) can readily be generalized to any odd dimension. In  $2n + 1$  dimensions it is of the form [10]:

$$S_{CS}^{2n+1}[A] = \int \mathcal{L}^{2n+1} \quad (6)$$

where  $\mathcal{L}^{2n+1}$  is a  $2n + 1$ -form defined by the requirement that its exterior derivative take the form:

$$d\mathcal{L}^{2n+1} = < (F[A])^n > \quad (7)$$

where  $F[A] = dA + \frac{1}{2}[A, A]$  is the field strength and the product of forms here denotes a wedge product: e.g.  $(F[A])^2 := F[A] \wedge F[A]$ , etc. The angle brackets  $< \dots >$  denote a symmetric, invariant  $n$ -linear form in the Lie algebra of  $G$ . It can be verified that the following Lagrangian density satisfies the above criterion:

$$\mathcal{L}^{2n+1} = (n+1) \sum_{i=0}^n \frac{n!}{(n+i+1)i!(n-i)!} < A^{2i+1} dA^{n-1} > \quad (8)$$

In contrast to 2+1 dimensions, the higher dimensional Chern-Simons theory does have local, physical degrees of freedom [11]. Moreover, the phase space is stratified into “layers” of different dimension. In the generic sector,

which has maximal phase space dimension, there are  $N - 2$  dynamical modes for a gauge group  $G$  of rank  $N$ .

Within the general class of theories described above, there is a subset that has special importance for the present discussion. Consider the case in which one has a gauge group  $\hat{G}$  that, rather than being semi-simple, is a direct product  $\hat{G} = G \times U(1)$ . Banados *et al* [11] showed that Chern-Simons theory in  $2n + 1$  dimensions possesses an algebra of surface charges that is isomorphic to the algebra for the  $WZW_{2n}$  model. This is a  $2n$  dimensional generalization of the  $WZW_2$  model. The action for the  $2n$  dimensional model is[12]:

$$S_{WZW_{(2n)}} = \frac{i}{4\pi} \int_M \omega^{n-1} \wedge \text{Tr}(h^{-1} \partial h \wedge h^{-1} \bar{\partial} h) + \frac{i}{12\pi} \int_{\partial M} \omega^{n-1} \wedge \text{Tr}(h^{-1} dh)^3 \quad (9)$$

where  $h$  is a field that takes its values in the group  $G$  and  $\omega$  is a Kahler form that, in holomorphic coordinates  $(z^\alpha, z^{\bar{\beta}})$ , takes the form:

$$\omega = (if_\pi^2/2) \omega_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}} \quad (10)$$

$\partial, \bar{\partial}$  denote partial derivatives with respect to the corresponding holomorphic coordinates  $z^\alpha, z^{\bar{\beta}}$ . The field equations that extremize the action Eq.(9) are:

$$\bar{\partial}(\omega^{n-1} \wedge h^{-1} \partial h) = 0 \quad (11)$$

or equivalently

$$\partial(\omega^{n-1} \wedge h^{-1} \bar{\partial} h) = 0 \quad (12)$$

In the following section we will restrict our attention to  $n = 2$ . We will show that for 4+1 dimensional Chern Simons gravity, the boundary action, dynamics and symmetry algebra are those of the  $WZW_4$  model. This latter model has been studied by a variety of authors [13]. Its field equations are equivalent to those of self-dual Yang-Mills theory in a particular gauge and the model is exactly solvable. Moreover, it was shown by Ketov [14], that the model is finite at one loop and it is speculated that it may be finite at all orders. The symmetry algebra for  $WZW_4$  is the so-called ‘‘two-toroidal Lie algebra’’:

$$\{Q^a(x), Q^b(y)\} = \frac{1}{2} f_{cd}^a g^{bc} Q^d \delta(x - y) + \frac{1}{2} \epsilon^{ijk} \omega_{ij} \partial_k g^{ab} \delta(x - y) \quad (13)$$

It is a generalization of the Kac-Moody algebra and has previously been studied by Mickelsson in the context of topologically massive Yang-Mills theory [15].

### 3 Boundary Dynamics 4+1 AdS Chern-Simons Gravity

We start with the Chern-Simons action in 4+1 dimensions with gauge group  $\hat{G} = \text{SO}(4, 2) \times U(1)$ . From Eq.(6) and Eq.(8) we have:

$$S_{CS}^5 = \int <\hat{A} \wedge d\hat{A} \wedge \hat{A}> + \frac{2}{5} <\hat{A}^5> + \frac{3}{2} <\hat{A}^3 d\hat{A}> \quad (14)$$

where  $\hat{A}$  is a one form that takes its values in the Lie algebra of  $\hat{G}$ . In a suitable basis it can be decomposed:  $\hat{A} = A^i J_i + a J_0$ , where  $J_a$ ,  $i = 1..15$  are the generators of  $\text{SO}(4, 2)$  and  $J_0$  is the generator corresponding to  $U(1)$ . In this basis we can define a symmetric trilinear form suitable for constructing the C-S action Eq.[14]:

$$\begin{aligned} \langle J_i J_j J_k \rangle &= \text{Tr}(J_i J_j J_k) \\ \langle J_i J_0 J_j \rangle &= \langle J_i J_j J_0 \rangle = \text{Tr}(J_i J_j) \\ \langle J_0 J_0 J_i \rangle &= \langle J_0 J_0 J_0 \rangle = 0 \end{aligned} \quad (15)$$

The action, expressed in terms of the fields  $A$  and  $a$  is:

$$S_{CS}^5[\hat{A}] = S_{CS}^5[A] + 3 \int_{M^5} \text{Tr}(A dA + \frac{2}{3} A^3) \omega, \quad (16)$$

The field equations are:

$$\text{Tr}((F \wedge F + 2F \wedge \omega) J_a) = 0 \quad (17)$$

$$\text{Tr}(F \wedge F) = 0 \quad (18)$$

As mentioned above the solution space for this theory is much richer than the 2+1 dimensional case. There are local dynamical degrees of freedom and the phase space splits into “strata” of different dimensions. We will focus on the “generic” sector of the theory, which has maximal number of degrees of freedom. This sector contains the physically relevant class of solutions:

$$\begin{aligned} F[A] &= 0 \\ \omega[a] &= \text{invertible but otherwise arbitrary} \end{aligned} \quad (19)$$

The reason that these solutions are interesting is that they correspond to AdS spacetime in the gravity theory. In order to make this connection explicit, consider the 5-metric in *bein* form:

$$ds_5^2 = \eta_{ab} e_\mu^a e_\nu^b dx^\mu dx^\nu \quad (20)$$

and define  $SO(4,2)$  Lie-algebra valued connection one form by:

$$A = \begin{bmatrix} w^a{}_b & e^a \\ -e_b & 0 \end{bmatrix} \quad (21)$$

where  $\omega^a{}_b$  is the spin connection for the frame fields. It can then be verified that the condition  $F[A] = 0$  in terms of the geometrical fields implies that the torsion vanishes and that the metric is locally AdS. It has been shown [17] that locally AdS black holes (analogous to the BTZ black hole) exist in 4+1 dimensions, so the Chern-Simons gravity theory under consideration does admit at least such black hole solutions. In the following we are interested only in the asymptotics, and we will therefore only impose the condition  $F[A] = 0$  at infinity. Whether or not the theory admits black hole solutions that are AdS only asymptotically is still an open question currently under consideration.

The plan for deriving the boundary dynamics is roughly the following. We start with the action (16) and require that on the boundary  $F[A] = 0$  while  $\omega[a]$  is a fixed, invertible, but otherwise arbitrary two form. Note that in contrast to most previous work, the boundary conditions that we are imposing are *gauge invariant*, so that the gauge potentials are fixed on the boundary only up to arbitrary gauge transformations. The next step is to figure out what boundary term must be added to the action in order to make the variational principle well defined. The result is that a suitable boundary term does exist if the gauge modes on the boundary obey the  $WZW_4$  field equations. Thus, as in the 2+1 dimensional theory, the gauge modes on the boundary obey dynamical equations, and are therefore physical. The corresponding boundary action is precisely that of  $WZW_4$ . We will now summarize the calculations that lead to this result. Details can be found in [9] and [18].

Consider the total action, including boundary term:

$$S_{tot} = S_{CS}^5[\hat{A}] + 3 \int_{\partial M} \text{Tr}(A_+ \wedge A_-) \quad (22)$$

where  $A_+ = A_\alpha dz^\alpha$  and  $A_- = A_{\bar{\alpha}} dz^{\bar{\alpha}}$  and  $(z^\alpha, z^{\bar{\alpha}})$  are holomorphic coordinates on the boundary  $\partial M$  of the five dimensional manifold  $M$ . Direct calculation reveals that the variation of the total action contains a piece in addition to the standard piece that vanishes when the bulk equations of motion are satisfied:

$$\delta S_{tot} = \int_M (\text{Eq. of Motion}) - 6 \int_{\partial M} (A_- \wedge \delta A_+) \wedge \omega \quad (23)$$

The second term in the above seems to imply that we have failed: the boundary term we have added does not totally eliminate boundary variations, and

according to standard lore, the variational principle for the bulk modes is still not well defined. However, we recall that our boundary conditions imply that  $A$  must be at least locally pure gauge on  $\partial M$ , so that

$$\begin{aligned} A_+|_{\text{bound}} &= (h^{-1}\partial_\alpha h)dz^\alpha \\ A_-|_{\text{bound}} &= (h^{-1}\partial_{\bar{\alpha}} h)dz^{\bar{\alpha}} \end{aligned} \quad (24)$$

and the boundary variation takes the form:

$$6 \int_{\partial M} \text{Tr}(\partial(h^{-1}\bar{\partial}h \wedge \omega)\delta h) \quad (25)$$

Thus the boundary term in the variation of the action vanishes if and only if the gauge modes  $h$  on the boundary satisfy the WZW4 equations of motion:

$$\partial(h^{-1}\bar{\partial}h \wedge \omega) = 0 \quad (26)$$

In order to verify that the proposed boundary term gives a consistent variational principle, we evaluate the full action for generic solutions to the bulk field equations, i.e. for flat connection  $A = h^{-1}dh$  and arbitrary  $a$ . The total action is not gauge invariant, and hence does not vanish for flat connections. Instead one gets:

$$S[h] = - \int_M \text{Tr}(h^{-1}dh)^3 \wedge \omega + 3 \int_{\partial M} \text{Tr}[(h^{-1}\partial h) \wedge (h^{-1}\bar{\partial}h)] \wedge \omega, \quad (27)$$

This is the WZW<sub>4</sub> action, whose variation yields Eq.(26) above.

Note that physical, gauge invariant quantities in the bulk are not restricted by these extra field equations, which only impose conditions on the boundary values of gauge transformations that can be performed on the potentials.

The decomposition into gauge invariant bulk modes and the physical gauge modes on the boundary can be made explicit using a parameterization first used in [19]. The parametrization consists of the usual Hodge decomposition for the abelian gauge potential:

$$a = a_h + d\lambda + \delta\beta \quad (28)$$

In the above,  $a_h$  is the harmonic part of the abelian potential  $a$ ,  $d$  is an exterior derivative,  $\delta$  is the corresponding co-derivative and  $\beta$  is a (gauge invariant) two form that determines the abelian field strength  $\omega$ , via the relationship:  $\omega = d\delta\beta$ . For the non-Abelian gauge potentials, we use a generalized Hodge decomposition:

$$A = \tilde{A} + \delta_{\tilde{A}}\alpha^L \quad (29)$$

where  $\alpha^L$  is a longitudinal two-form,  $\tilde{A}$  is a flat connection that can itself be parametrized in terms of global gauge transformations  $h$ :

$$\tilde{A} = h^{-1}\theta h + h^{-1}dh \quad (30)$$

with  $\theta$  its topologically non-trivial part that cannot be globally gauge transformed to zero. Moreover,  $\delta_{\tilde{A}}$  is a covariant, co-exact derivative with respect to the flat connection  $\tilde{A}$ . For details, and motivation for this parametrization, see [19]. The net effect of the above is to express an arbitrary field configuration in terms of a gauge field  $h$ , a longitudinal, gauge covariant two form  $\alpha^L$ , and a topologically non-trivial flat connection  $\theta$ . In terms of this parametrization, the partition function looks like:

$$\begin{aligned} Z &= \int \frac{\mathcal{D}A}{V_G} \frac{\mathcal{D}a}{V_{U(1)}} e^{iS_{tot}} \\ &= \int \mathcal{D}\theta J[\theta] Z_{WZW_4}[h_B, \theta] Z_{CS}[\beta, \alpha, \theta] \end{aligned} \quad (31)$$

Thus the partition function splits into a factor that is essentially the Chern-Simons partition function for the bulk modes with the given boundary conditions, and a factor that is the  $WZW_4$  partition function for the boundary modes. There is in addition an integral, with appropriate measure, over the topologically inequivalent flat connections,  $\theta$  that the spacetime admits for the given boundary conditions. A detailed analysis of this partition function will be presented in [18]

## 4 Conclusion

We have shown that asymptotically AdS boundary conditions in 4+1 dimensional Chern-Simons gravity lead to a  $WZW_4$  boundary action, in analogy with what happens in 2+1 dimensions. This talk started with the conjecture that such boundary terms will play an important role in our ultimate understanding of black hole entropy. The hope was to generalize the analysis of Carlip [1] or Strominger [6] to dimensions higher than three. However, all we have so far is the boundary theory in 4+1 dimensions. We do not as yet know how to count states, since the representation theory of the toroidal Lie algebras is not well known, although some results have been reported by Billig [16]. The other thing we would like to do is to extend the analysis to higher dimensions. A preliminary analysis seems to indicate that the boundary action is not precisely  $WZW_{2n}$  but this is currently under investigation as well.

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